

Asymptotic formulae of two divergent bilateral basic hypergeometric series

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Abstract

We provide new formulae for the degenerations of the bilateral basic hypergeometric function ${}_1\psi_1(a; b; q, z)$ with using the q -Borel-Laplace transformation. These are thought of as the first step to construct connection formulae of q -difference equation for ${}_1\psi_1(a; b; q, z)$. Moreover, we show that our formulae have the $q \rightarrow 1 - 0$ limit.

1 Introduction

In this paper, we give two asymptotic formulae for the bilateral basic hypergeometric series,

$${}_1\psi_1(0; b; q, x) := \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} x^n$$

and

$${}_1\psi_0(a; -; q, x) := \sum_{n \in \mathbb{Z}} (a; q)_n \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{-1} x^n$$

from the viewpoint of the connection problems on q -difference equations. Here, $(a; q)_n$ is the q -shifted factorial defined by

$$(a; q)_n := \begin{cases} 1, & n = 0, \\ (1-a)(1-aq) \cdots (1-aq^{n-1}), & n \geq 1, \\ [(1-aq^{-1})(1-aq^{-2}) \cdots (1-aq^n)]^{-1}, & n \leq -1. \end{cases}$$

Notice that the q -shifted factorial is the q -analogue of the shifted factorial

$$(\alpha)_n = \alpha\{\alpha+1\} \cdots \{\alpha+(n-1)\}.$$

Moreover, $(a; q)_\infty := \lim_{n \rightarrow \infty} (a; q)_n$ and we use the shorthand notation

$$(a_1, a_2, \dots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

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Connection problems on q -difference equations are originally studied by G. D. Birkhoff [1]. At first, we review the connection problems on second order q -difference equations of the form

$$(a_0 + b_0x)u(q^2x) + (a_1 + b_1x)u(qx) + (a_2 + b_2x)u(x) = 0, \quad (1)$$

where $a_0a_2b_0b_2 \neq 0$. Let $u_1(x)$, $u_2(x)$ be independent solutions of equation (1) around the origin and let $v_1(x)$, $v_2(x)$ be those around infinity. We take suitable analytic continuation of $u_1(x)$ and $u_2(x)$ [2]. Then we obtain the connection formulae in the following matrix form:

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} C_{11}(x) & C_{12}(x) \\ C_{21}(x) & C_{22}(x) \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}.$$

Here, functions $C_{jk}(x)$, ($j, k = 1, 2$) are q -periodic and unique valued, namely, elliptic functions. G. N. Watson gave the first example of the connection formula. He showed a connection formula for the (unilateral) basic hypergeometric series

$${}_2\varphi_1(a, b; c; q, x) := \sum_{n \geq 0} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} x^n \quad (2)$$

The series (2) satisfies the second order q -difference equation

$$(c - abqx)u(q^2x) - \{c + q - (a + b)qx\}u(qx) + q(1 - x)u(x) = 0 \quad (3)$$

around the origin. Equation (3) also has solutions around infinity as follows:

$$v_1(x) = \frac{(ax, q/ax; q)_\infty}{(x, q/x; q)_\infty} {}_2\varphi_1\left(a, \frac{aq}{c}; \frac{aq}{b}; q, \frac{cq}{abx}\right), \quad v_2(x) = \frac{(bx, q/bx; q)_\infty}{(x, q/x; q)_\infty} {}_2\varphi_1\left(b, \frac{bq}{c}; \frac{bq}{a}; q, \frac{cq}{abx}\right).$$

The connection formula by Watson [3] is

$$\begin{aligned} u_1(x) = & \frac{(b, c/a; q)_\infty}{(c, b/a; q)_\infty} \frac{(-ax, -q/ax; q)_\infty}{(-x, -q/x; q)_\infty} \frac{(x, q/x; q)_\infty}{(ax, q/ax; q)_\infty} v_1(x) \\ & + \frac{(a, c/b; q)_\infty}{(c, a/b; q)_\infty} \frac{(-bx, -q/bx; q)_\infty}{(-x, -q/x; q)_\infty} \frac{(x, q/x; q)_\infty}{(bx, q/bx; q)_\infty} v_2(x). \end{aligned}$$

We remark that each connection coefficient is given by the elliptic function.

In equation (1), if we assume that $a_0a_2b_0b_2 = 0$, some power series which appear in formal solutions may be divergent. Therefore, we should take a suitable resummation of a divergent series. J.-P. Ramis and C. Zhang introduced the q -Borel-Laplace resummation method to study the connection problems. The q -Borel-Laplace transformation is given as follows:

1. We assume that $f(x) = \sum_{n \geq 0} a_n x^n$ is a formal power series. The q -Borel transformation of the first kind \mathcal{B}_q^+ is given by

$$(\mathcal{B}_q^+ f)(\xi) := \sum_{n \geq 0} a_n q^{\frac{n(n-1)}{2}} \xi^n. \quad (4)$$

We denote $\varphi_f(\xi) = (\mathcal{B}_q^+ f)(\xi)$. If $f(x)$ is a convergent series, then $\varphi_f(\xi)$ is an entire function.

2. We fix $\lambda \in \mathbb{C}^* \setminus q^{\mathbb{Z}}$. For any entire function $\varphi(\xi)$, the q -Laplace transformation of the first kind $\mathcal{L}_{q,\lambda}^+$ [4] is given by

$$\left(\mathcal{L}_{q,\lambda}^+\varphi\right)(x) := \frac{1}{1-q} \int_0^{\lambda\infty} \frac{\varphi(\xi)}{\theta_q\left(\frac{\xi}{x}\right)} \frac{d_q\xi}{\xi} = \sum_{n \in \mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)}, \quad (5)$$

where

$$\int_0^{\lambda\infty} f(t) d_q t := (1-q)\lambda \sum_{n \in \mathbb{Z}} f(\lambda q^n) q^n$$

is Jackson's q -integral on $(0, \lambda\infty)$ [2].

Thanks to these resummation methods, Zhang gave a connection formula for a divergent series ${}_2\varphi_0(a, b; -; q, x)$ [4]. Morita also gave connection formulae for a divergent series ${}_2\varphi_0(0, 0; -; q, x)$ [5] and some unilateral divergent series [6] by these transformations.

But when we consider the connection formulae for the *bilateral* series, the connection problems are not so clear. Though L. J. Slater gave a relation between the bilateral series ${}_r\psi_r$ in [7], other relations are not known well. The main aim of this paper is to give (connection) formulae between divergent bilateral series and convergent unilateral series by q -Borel-Laplace transformations in Section 3. In the last section, we also give the classical limit $q \rightarrow 1 - 0$ of our new formulae.

We are closing this section with commenting on relationship with physics. We proved summation formulae of ${}_1\psi_1$ in [8] which are politic generalizations of the ones found from the study of physics called Abelian mirror symmetry in three-dimensional supersymmetric gauge theories on the unorientable manifold $\mathbb{RP}^2 \times S^1$ [9, 10]. We hope our new formulae here could open up the class of connection formulae and provide prominent insight into physics.

2 Notation

In this section, we declare basic notation. In the following, we assume that $0 < |q| < 1$. The bilateral basic hypergeometric series with the base q is defined by

$${}_r\psi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) := \sum_{n \in \mathbb{Z}} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{s-r} z^n. \quad (6)$$

This series diverges for $z \neq 0$ if $s < r$ and converges for $|b_1 \dots b_s / a_1 \dots a_r| < |z| < 1$ if $r = s$ (refer to [2] for more details). Further, the series (6) is the q -analogue of the bilateral hypergeometric function

$${}_rH_s(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; z) := \sum_{n \in \mathbb{Z}} \frac{(\alpha_1, \dots, \alpha_r)_n}{(\beta_1, \dots, \beta_s)_n} z^n,$$

which is the bilateral extension of the generalized hypergeometric function

$${}_rF_s(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; z) := \sum_{n \geq 0} \frac{(\alpha_1, \dots, \alpha_r)_n}{(\beta_1, \dots, \beta_s)_n} \frac{z^n}{n!}.$$

Provided $\Re(\beta_1 + \dots + \beta_r - \alpha_1 - \dots - \alpha_r) > 1$, D'Alembert's ratio test could verify that ${}_rH_r$ converges only for $|z| = 1$ [7]. For later use, we would like to write down Ramanujan's summation formula given by S. Ramanujan [11],

$${}_1\psi_1(a; b; q, z) = \sum_{n \in \mathbb{Z}} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty} \quad (7)$$

with $|b/a| < |z| < |1|$. Ramanujan's summation formula is considered as the bilateral extension of the q -binomial theorem [2]

$$\sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty} \quad (8)$$

for $|z| < 1$. The q -binomial theorem was derived by Cauchy [12], Heine [13], and many mathematicians.

One of the q -exponential functions is defined as

$$E_q(z) := {}_0\varphi_0(-; -; q, -z) = \sum_{n \geq 0} \frac{1}{(q; q)_n} (-1)^n q^{\frac{n(n-1)}{2}} (-z)^n, \quad (9)$$

which can be rewritten by the infinite product expression

$$E_q(z) = (-z; q)_\infty \quad (10)$$

with $|z| < 1$. We note that the limit $q \rightarrow 1 - 0$ of this q -exponential is actually the standard exponential

$$\lim_{q \rightarrow 1-0} E_q(z(1-q)) = e^z. \quad (11)$$

The q -gamma function $\Gamma_q(z)$ is given by

$$\Gamma_q(z) := \frac{(q; q)_\infty}{(q^z; q)_\infty} (1-q)^{1-z}. \quad (12)$$

The $q \rightarrow 1 - 0$ limit of $\Gamma_q(z)$ reproduces the gamma function [2]

$$\lim_{q \rightarrow 1-0} \Gamma_q(z) = \Gamma(z). \quad (13)$$

The theta function of Jacobi with the base q which we will use is given by

$$\theta_q(z) := \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} z^n, \quad \forall z \in \mathbb{C}^*. \quad (14)$$

With this definition, Jacobi's triple product identity is expressed by

$$\theta_q(z) = (q, -z, -q/z; q)_\infty. \quad (15)$$

The theta function has the inversion formula

$$\theta_q(z) = \theta_q(q/z), \quad (16)$$

and satisfies the q -difference equation

$$\theta_q(zq^k) = z^{-k} q^{-\frac{k(k-1)}{2}} \theta_q(z). \quad (17)$$

In our study, the following proposition about the theta function [14, 15] is useful to consider the $q \rightarrow 1-0$ limit of our formulae in Section 4.

Proposition 1. *For any $z \in \mathbb{C}^*$ ($-\pi < \arg z < \pi$), we have*

$$\lim_{q \rightarrow 1-0} \frac{\theta_q(q^\beta z)}{\theta_q(q^\alpha z)} = z^{\alpha-\beta}. \quad (18)$$

and

$$\lim_{q \rightarrow 1-0} \frac{\theta_q\left(\frac{q^\alpha z}{(1-q)}\right)}{\theta_q\left(\frac{q^\beta z}{(1-q)}\right)} (1-q)^{\beta-\alpha} = z^{\beta-\alpha}. \quad (19)$$

We also use the following limiting formula of the ratio of the q -shifted factorial [2]:

$$\lim_{q \rightarrow 1-0} \frac{(zq^\alpha; q)_\infty}{(z; q)_\infty} = (1-z)^{-\alpha}, \quad |z| < 1. \quad (20)$$

The crucial ingredients for our new formulae are the q -Borel-Laplace transformation. In the rest of the paper, we concentrate on the sequence of the action of the q -Borel-Laplace transformation on degenerations of the divergent series ${}_1\psi_1^{\deg}$,

$${}_1\psi_1^{\deg}(x) \xrightarrow{\mathcal{B}_q^+} \psi(\xi) \xrightarrow{\mathcal{L}_{q,\lambda}^+} \tilde{\psi}_\lambda(x).$$

We remark that λ -dependence on $\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ f$ vanishes, i.e., $\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ f = f$ if $f(x)$ is a convergent series.

3 Degenerations of the bilateral basic hypergeometric series

In this section, we will provide new convergent series obtained by acting the q -Borel-Laplace transformation on degenerations of the bilateral basic hypergeometric series ${}_1\psi_1(a; b; q, x)$ which satisfies the first order q -difference equation

$$\left(\frac{b}{q} - ax\right) u(qx) + (x-1)u(x) = 0. \quad (21)$$

We consider two different degenerations of equation (21).

1. Degeneration A

In the equation (21), if we take the limit $a \rightarrow 0$, we obtain the following equation:

$$\frac{b}{q}\tilde{u}(x) + (x-1)\tilde{u}(x) = 0. \quad (22)$$

The bilateral series solution is given by

$$\tilde{u}(x) = {}_1\psi_1(0; b; q, x). \quad (23)$$

We remark that the series is a divergent series around the origin. We can also find a unilateral series solution *around the origin* as follows:

$$\tilde{v}(x) = \frac{\theta_q(bx)}{\theta_q(qx)} {}_1\varphi_0(0; -; q, x). \quad (24)$$

2. Degeneration B

In the equation (21), if we put $x \mapsto bx$ and take the limit $b \rightarrow \infty$, we obtain another equation as follows:

$$\left(\frac{1}{q} - ax\right)\hat{u}(qx) + x\hat{u}(x) = 0. \quad (25)$$

The bilateral series solution is

$$\hat{u}(x) = {}_1\psi_0(a; -; q, x). \quad (26)$$

We remark that the solution $\hat{u}(x)$ contains a divergent series around the origin. We also find the unilateral basic hypergeometric series solution around infinity is

$$\hat{v}(x) = \frac{\theta_q(ax)}{\theta_q(x)} {}_1\varphi_0\left(0; -; q, \frac{1}{ax}\right). \quad (27)$$

In the following subsection, we consider asymptotic formulae for these two divergent series.

3.1 Degeneration A

We here deal with the divergent series (23),

$${}_1\psi_1(0; b; q, x) = \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} x^n. \quad (28)$$

As preparation, let us show the following formula:

Lemma 1. *For any $x \in \mathbb{C}^*$, we have*

$${}_0\psi_1(-; b; q, x) = \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(-x)}{\theta_q(-\frac{qx}{b})} \left(\frac{qx}{b}; q\right)_\infty. \quad (29)$$

Proof. Firstly, we scale $x \rightarrow x/a$ in the bilateral basic hypergeometric function,

$$\begin{aligned} {}_1\psi_1(a; b; q, x/a) &= \sum_{n \in \mathbb{Z}} \frac{(a; q)_n}{(b; q)_n} \frac{x^n}{a^n} \\ &= \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} \frac{(1-a)(1-aq) \cdots (1-aq^n)}{a^n} x^n \\ &= \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} \left(\frac{1}{a} - 1 \right) \left(\frac{1}{a} - q \right) \cdots \left(\frac{1}{a} - q^n \right) x^n. \end{aligned}$$

Then, taking the limit $a \rightarrow \infty$ gives

$$\lim_{a \rightarrow \infty} {}_1\psi_1(a; b; q, x/a) = \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} (-1)^n q^{\frac{n(n-1)}{2}} x^n = {}_0\psi_1(-; b; q, x).$$

On the other hand, we repeat the same process into Ramanujan's summation formula (7),

$$\lim_{a \rightarrow \infty} {}_1\psi_1(a; b; q, x/a) = \lim_{a \rightarrow \infty} \frac{(q, \frac{b}{a}, x, \frac{q}{x}; q)_\infty}{(b, \frac{q}{a}, \frac{x}{a}, \frac{b}{x}; q)_\infty} = \frac{(q, x, \frac{q}{x}; q)_\infty}{(b, \frac{b}{x}; q)_\infty}.$$

Therefore,

$$\begin{aligned} {}_0\psi_1(-; b; q, x) &= \frac{(q, x, \frac{q}{x}; q)_\infty}{(b, \frac{b}{x}; q)_\infty} \\ &= \frac{(q; q)_\infty}{(b; q)_\infty} \frac{(x, \frac{q}{x}, q; q)_\infty}{(\frac{b}{x}, \frac{qx}{b}, q; q)_\infty} \left(\frac{qx}{b}; q \right)_\infty \\ &= \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(-x)}{\theta_q(-\frac{qx}{b})} \left(\frac{qx}{b}; q \right)_\infty, \end{aligned}$$

which we actually would like to show. \square

The main purpose is to apply the q -Borel-Laplace transformation to the divergent series ${}_1\psi_1(0; b; q, x)$. As a result, we can find the following convergent series for the degeneration $a \rightarrow 0$ of ${}_1\psi_1$:

Theorem 1. *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have*

$$\tilde{\psi}_\lambda^A(x) = \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(\lambda) \theta_q\left(\frac{\lambda q}{bx}\right)}{\theta_q\left(\frac{q\lambda}{b}\right) \theta_q\left(\frac{\lambda}{x}\right)} {}_1\varphi_0(0; -; q, x),$$

where $\tilde{\psi}_\lambda^A(x) := \left(\mathcal{L}_{q, \lambda}^+ \circ \mathcal{B}_q^+ {}_1\psi_1 \right) (0; b; q, x)$.

Proof. The q -Borel transformation (4) of ${}_1\psi_1(0; b; q, x)$ provides

$$\begin{aligned} \psi_A(\xi) &:= \left(\mathcal{B}_q^+ {}_1\psi_1 \right) (0; b; q, x) \\ &= \sum_{n \in \mathbb{Z}} \frac{1}{(b; q)_n} q^{\frac{n(n-1)}{2}} \xi^n \\ &= {}_0\psi_1(-; b; q, -\xi). \end{aligned}$$

Actually, ${}_0\psi_1$ has the degenerated version of the Ramanujan's summation formula shown in Lemma 1, that is,

$$\begin{aligned}\psi_A(\xi) &= \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(\xi)}{\theta_q\left(\frac{q\xi}{b}\right)} \left(-\frac{q\xi}{b}; q\right)_\infty \\ &= \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(\xi)}{\theta_q\left(\frac{q\xi}{b}\right)} E_q\left(\frac{q\xi}{b}\right).\end{aligned}$$

Then, we take the q -Laplace transformation (5) to $\psi_A(\xi)$,

$$\begin{aligned}\tilde{\psi}_\lambda^A(x) &:= \left(\mathcal{L}_{q,\lambda}^+ \psi_A\right)(\xi) \\ &= \sum_{n \in \mathbb{Z}} \frac{{}_0\psi_1(-; b; q, -\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)} \\ &= \frac{(q; q)_\infty}{(b; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{\theta_q(\lambda q^n)}{\theta_q\left(\frac{q\lambda q^n}{b}\right)} \frac{1}{\theta_q\left(\frac{\lambda q^n}{x}\right)} \sum_{m \geq 0} \frac{(-1)^m q^{\frac{m(m-1)}{2}}}{(q; q)_m} \left(-\frac{q\lambda q^n}{b}\right)^m \\ &= \frac{(q; q)_\infty}{(b; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{\lambda^{-n} q^{-\frac{n(n-1)}{2}} \theta_q(\lambda)}{\left(\frac{q\lambda}{b}\right)^{-n} q^{-\frac{n(n-1)}{2}} \theta_q\left(\frac{q\lambda}{b}\right)} \frac{1}{\left(\frac{\lambda}{x}\right)^{-n} q^{-\frac{n(n-1)}{2}} \theta_q\left(\frac{\lambda}{x}\right)} \sum_{m \geq 0} \frac{(-1)^m q^{\frac{m(m-1)}{2}}}{(q; q)_m} \left(-\frac{q\lambda q^n}{b}\right)^m \\ &= \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(\lambda)}{\theta_q\left(\frac{q\lambda}{b}\right) \theta_q\left(\frac{\lambda}{x}\right)} \sum_{n \in \mathbb{Z}} \sum_{m \geq 0} \left(\frac{\lambda q}{bx}\right)^{n+m} q^{\frac{(n+m)(n+m-1)}{2}} \frac{1}{(q; q)_m} x^m \\ &= \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(\lambda)}{\theta_q\left(\frac{q\lambda}{b}\right) \theta_q\left(\frac{\lambda}{x}\right)} \sum_{N \in \mathbb{Z}} \left(\frac{\lambda q}{bx}\right)^N q^{\frac{N(N-1)}{2}} \sum_{m \geq 0} \frac{1}{(q; q)_m} x^m \\ &= \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(\lambda)}{\theta_q\left(\frac{q\lambda}{b}\right) \theta_q\left(\frac{\lambda}{x}\right)} \theta_q\left(\frac{\lambda q}{bx}\right) {}_1\varphi_0(0; -; q, x).\end{aligned}$$

The final expression is obviously a convergent series with $x \in \mathbb{C}^* \setminus [-\lambda; q]$ as expected. \square

Corollary 1. *By Theorem 1 and (24), we obtain the following relation:*

$$\tilde{\psi}_\lambda^A(x) = \frac{(q; q)_\infty}{(b; q)_\infty} \frac{\theta_q(\lambda)}{\theta_q\left(\frac{q\lambda}{b}\right)} \frac{\theta_q\left(\frac{bx}{\lambda}\right)}{\theta_q\left(\frac{qx}{\lambda}\right)} \frac{\theta_q(qx)}{\theta_q(bx)} \tilde{v}(x) =: C_A(x) \tilde{v}(x).$$

We remark that the function $C_A(x)$ is an elliptic function, namely, q -periodic and unique valued.

3.2 Degeneration B

Let us turn to divergent series (26),

$${}_1\psi_0(a; -; q, x) = \sum_{n \in \mathbb{Z}} (a; q)_n (-1)^n q^{-\frac{n(n-1)}{2}} x^n. \quad (30)$$

Applying the q -Borel-Laplace transformation to ${}_1\psi_0(a; -; q, x)$ can bring us to the following conclusion:

Theorem 2. For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have

$$\tilde{\psi}_\lambda^B(x) = \frac{(q; q)_\infty}{(\frac{q}{a}; q)_\infty} \frac{\theta_q(a\lambda)\theta_q(\frac{aqx}{\lambda})}{\theta_q(\lambda)\theta_q(\frac{qx}{\lambda})} {}_1\varphi_0\left(0; -, q, \frac{1}{ax}\right),$$

where $\tilde{\psi}_\lambda^B(x) := \left(\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ \psi_0\right)(a; -, q, x)$.

Proof. We perform at first the q -Borel transformation (4),

$$\begin{aligned} \psi_B(\xi) &:= (\mathcal{B}_q^+ \psi_0)(a; -, q, x) \\ &= \sum_{n \in \mathbb{Z}} (a; q)_n (-\xi)^n \\ &= {}_1\psi_1(a; 0; q, -\xi). \end{aligned}$$

Then, it is clear that Ramanujan's summation formula directly works on $\psi_B(\xi)$ as

$$\begin{aligned} \psi_B(\xi) &= \frac{(q, -a\xi, -\frac{q}{a\xi}; q)_\infty}{(\frac{q}{a}, -\xi; q)_\infty} \\ &= \frac{(q; q)_\infty}{(\frac{q}{a}; q)_\infty} \frac{\theta_q(a\xi)}{\theta_q(\xi)} E_q\left(\frac{q}{\xi}\right). \end{aligned}$$

Finally, the q -Laplace transformation (5) of $\psi_B(\xi)$ results in

$$\begin{aligned} \tilde{\psi}_\lambda^B(x) &:= \left(\mathcal{L}_{q,\lambda}^+ \psi_B(\xi)\right)(\xi) \\ &= \sum_{n \in \mathbb{Z}} \frac{{}_1\psi_1(a; 0; q, -\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)} \\ &= \frac{(q; q)_\infty}{(\frac{q}{a}; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{\theta_q(a\lambda q^n)}{\theta_q(\lambda q^n)} \frac{1}{\theta_q\left(\frac{\lambda q^n}{x}\right)} \sum_{m \geq 0} \frac{(-1)^m q^{\frac{m(m-1)}{2}}}{(q; q)_m} \left(-\frac{q}{\lambda q^n}\right)^m \\ &= \frac{(q; q)_\infty}{(\frac{q}{a}; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(a\lambda)^{-n} q^{-\frac{n(n-1)}{2}} \theta_q(a\lambda)}{\lambda^{-n} q^{-\frac{n(n-1)}{2}} \theta_q(\lambda)} \frac{1}{\left(\frac{\lambda}{x}\right)^{-n} q^{-\frac{n(n-1)}{2}} \theta_q\left(\frac{\lambda}{x}\right)} \sum_{m \geq 0} \frac{(-1)^m q^{\frac{m(m-1)}{2}}}{(q; q)_m} \left(-\frac{q}{\lambda q^n}\right)^m \\ &= \frac{(q; q)_\infty}{(\frac{q}{a}; q)_\infty} \frac{\theta_q(a\lambda)}{\theta_q(\lambda)\theta_q\left(\frac{\lambda}{x}\right)} \sum_{n \in \mathbb{Z}} \sum_{m \geq 0} \left(\frac{\lambda}{ax}\right)^{n-m} q^{\frac{(n-m)(n-m-1)}{2}} \frac{1}{(q; q)_m} \left(\frac{1}{ax}\right)^m \\ &= \frac{(q; q)_\infty}{(\frac{q}{a}; q)_\infty} \frac{\theta_q(a\lambda)}{\theta_q(\lambda)\theta_q\left(\frac{\lambda}{x}\right)} \theta_q\left(\frac{\lambda}{ax}\right) {}_1\varphi_0\left(0; -, q, \frac{1}{ax}\right) \\ &= \frac{(q; q)_\infty}{(\frac{q}{a}; q)_\infty} \frac{\theta_q(a\lambda)}{\theta_q(\lambda)\theta_q\left(\frac{qx}{\lambda}\right)} \theta_q\left(\frac{aqx}{\lambda}\right) {}_1\varphi_0\left(0; -, q, \frac{1}{ax}\right), \end{aligned}$$

where in the last line the inversion formula (16) of the theta function is used. Therefore, we find the desired convergent series with $x \in \mathbb{C}^* \setminus [-\lambda; q]$. \square

Corollary 2. By Theorem 2 and (27), we obtain the following relation

$$\tilde{\psi}_\lambda^B(x) = \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{\theta_q(a\lambda)}{\theta_q(\lambda)} \frac{\theta_q\left(\frac{aqx}{\lambda}\right)}{\theta_q\left(\frac{qx}{\lambda}\right)} \frac{\theta_q(x)}{\theta_q(ax)} \hat{v}(x) =: C_B(x) \hat{v}(x).$$

We remark that the function $C_B(x)$ is an elliptic function, namely, q -periodic and unique valued.

4 The $q \rightarrow 1 - 0$ limit of the degenerations

We consider the limit $q \rightarrow 1 - 0$ of our formulae obtained in the previous section. Before showing those limiting formula, we also focus on the limit $q \rightarrow 1 - 0$ of the q -binomial theorem in the following *linear sum form* [9]:

$${}_1\varphi_0(a; -; q, x) = {}_2\varphi_1(a, aq; q; q^2, x^2) + x \frac{(a, q^3; q^2)_\infty}{(aq^2, q; q^2)_\infty} {}_2\varphi_1(aq, aq^2; q^3; q^2, x^2). \quad (31)$$

Proposition 2. *We put $a = q^\alpha$ in (31). Then, for any $x \in \mathbb{C}^*$, we have*

$${}_1F_0(\alpha; -; x) = {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha}{2} + \frac{1}{2}; \frac{1}{2}; x^2\right) + \alpha x {}_2F_1\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{\alpha}{2} + 1; \frac{3}{2}; x^2\right).$$

We can straightforwardly derive this sum expression and also verify this directly by decomposing the series expansion of ${}_1F_0(\alpha; -; x)$ into the summations over even and odd integers. In addition, we consider the limit $q \rightarrow 1 - 0$ of the special case $a = 0$ of (31):

$${}_1\varphi_0(0; -; q, x) = {}_2\varphi_1(0, 0; q; q^2, x^2) + x \frac{(q^3; q^2)_\infty}{(q; q^2)_\infty} {}_2\varphi_1(0, 0; q^3; q^2, x^2). \quad (32)$$

Proposition 3. *For any $x \in \mathbb{C}^*$, we have*

$${}_0F_0(-; -; 2x) = {}_0F_1\left(-; \frac{1}{2}; x^2\right) + 2x {}_0F_1\left(-; \frac{3}{2}; x^2\right).$$

Proof. We put $x \mapsto (1 - q^2)x$ in (32) and then implement the limit $q \rightarrow 1 - 0$. The left-hand side of (32) converges to the exponential function,

$$\lim_{q \rightarrow 1-0} \sum_{n \geq 0} \frac{(1 - q^2)^n}{(q; q)_n} x^n = \lim_{q \rightarrow 1-0} \sum_{n \geq 0} \frac{(1 - q)^n}{(q; q)_n} ((1 + q)x)^n = {}_0F_0(-; -; 2x) = e^{2x}.$$

On the other hand, the limit $q \rightarrow 1 - 0$ of the right-hand side with $x \mapsto (1 - q^2)x$ reduces to

$$\begin{aligned} & \lim_{q \rightarrow 1-0} \left\{ \sum_{n \geq 0} \frac{(1 - q^2)^{2n}}{(q; q^2)_n (q^2; q^2)_n} x^{2n} + (1 - q^2)x \frac{(q^3; q^2)_\infty}{(q; q^2)_\infty} \sum_{n \geq 0} \frac{(1 - q^2)^{2n}}{(q^3; q^2)_n (q^2; q^2)_n} x^{2n} \right\} \\ &= \lim_{q \rightarrow 1-0} \sum_{n \geq 0} \frac{(1 - q^2)^n (1 - q^2)^n}{(q; q^2)_n (q^2; q^2)_n} x^{2n} \\ & \quad + x \lim_{q \rightarrow 1-0} \frac{(q^2; q^2)_\infty}{((q^2)^{\frac{1}{2}}; q^2)_\infty} (1 - q^2)^{\frac{1}{2}} \times \frac{((q^2)^{\frac{3}{2}}; q^2)_\infty}{(q^2; q^2)_\infty} (1 - q^2)^{\frac{1}{2}} \sum_{n \geq 0} \frac{(1 - q^2)^n (1 - q^2)^n}{(q^3; q^2)_n (q^2; q^2)_n} x^{2n} \\ &= \lim_{q \rightarrow 1-0} \sum_{n \geq 0} \frac{(1 - q^2)^n (1 - q^2)^n}{(q; q^2)_n (q^2; q^2)_n} x^{2n} + x \lim_{q \rightarrow 1-0} \frac{\Gamma_{q^2}(\frac{1}{2})}{\Gamma_{q^2}(\frac{3}{2})} \sum_{n \geq 0} \frac{(1 - q^2)^n (1 - q^2)^n}{(q^3; q^2)_n (q^2; q^2)_n} x^{2n} \\ &= \sum_{n \geq 0} \frac{1}{(\frac{1}{2})_n n!} x^{2n} + x \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \sum_{n \geq 0} \frac{1}{(\frac{3}{2})_n n!} x^{2n} \\ &= {}_0F_1\left(-; \frac{1}{2}; x^2\right) + 2x {}_0F_1\left(-; \frac{3}{2}; x^2\right). \end{aligned}$$

Therefore, we obtain the conclusion. \square

4.1 On the degeneration A

Let us consider the limit $q \rightarrow 1 - 0$ of Theorem 1. We can find the well-defined limiting formula by scaling $x \mapsto (1 - q)x$.

Theorem 3. *For any $x \in \mathbb{C}^*$,*

$$\lim_{q \rightarrow 1-0} \tilde{\psi}_\lambda^A((1-q)x) = \Gamma(\beta)x^{1-\beta}e^x. \quad (33)$$

Proof. We put $x \mapsto (1-q)x$ and $b \mapsto q^\beta$ in Theorem 1 to obtain the limit. Then, we have

$$\begin{aligned} \tilde{\psi}_\lambda^A((1-q)x) &= \frac{(q; q)_\infty}{(q^\beta; q)_\infty} \frac{\theta_q(\lambda)}{\theta_q(q^{1-\beta}\lambda)} \frac{\theta_q\left(\frac{q^{1-\beta}}{(1-q)} \frac{\lambda}{x}\right)}{\theta_q\left(\frac{1}{(1-q)} \frac{\lambda}{x}\right)} \sum_{n \geq 0} \frac{(1-q)^n}{(q; q)_n} x^n \\ &= \frac{(q; q)_\infty}{(q^\beta; q)_\infty} (1-q)^{1-\beta} \frac{\theta_q(\lambda)}{\theta_q(q^{1-\beta}\lambda)} \frac{\theta_q\left(\frac{q^{1-\beta}}{(1-q)} \frac{\lambda}{x}\right)}{\theta_q\left(\frac{1}{(1-q)} \frac{\lambda}{x}\right)} (1-q)^{\beta-1} \sum_{n \geq 0} \frac{(1-q)^n}{(q; q)_n} x^n \\ &= \Gamma_q(\beta) \frac{\theta_q(\lambda)}{\theta_q(q^{1-\beta}\lambda)} \frac{\theta_q\left(\frac{q^{1-\beta}}{(1-q)} \frac{\lambda}{x}\right)}{\theta_q\left(\frac{1}{(1-q)} \frac{\lambda}{x}\right)} (1-q)^{\beta-1} \sum_{n \geq 0} \frac{(1-q)^n}{(q; q)_n} x^n \end{aligned} \quad (34)$$

Now, applying (19) in Proposition 1 to equation (34), we have

$$\lim_{q \rightarrow 0} \tilde{\psi}_\lambda^A((1-q)x) = \Gamma(\beta) \times (\lambda)^{1-\beta} \times \left(\frac{\lambda}{x}\right)^{\beta-1} \sum_{n \geq 0} \frac{1}{n!} x^n = \Gamma(\beta)(x)^{1-\beta}e^x,$$

which is the consistent limiting formula of Theorem 1 as $q \rightarrow 1 - 0$. \square

4.2 On the degeneration B

Finally, we treat with Theorem 2 in the limit $q \rightarrow 1 - 0$. With rescaling $x \mapsto x/(1-q)$, the limit $q \rightarrow 1 - 0$ of Theorem 2 turns out to be the following formula:

Theorem 4. *For any $x \in \mathbb{C}^*$,*

$$\lim_{q \rightarrow 1-0} \tilde{\psi}_\lambda^B\left(\frac{x}{1-q}\right) = \Gamma(1-\alpha)x^{-\alpha}e^{\frac{1}{x}}. \quad (35)$$

Proof. We put $x \mapsto x/(1-q)$ and $a \mapsto q^\alpha$ in Theorem 2 to consider the limit $q \rightarrow 1 - 0$. Then, we have

$$\begin{aligned} \tilde{\psi}_\lambda^B\left(\frac{x}{1-q}\right) &= \frac{(q; q)_\infty}{(q^{1-\alpha}; q)_\infty} \frac{\theta_q(q^\alpha\lambda)}{\theta_q(\lambda)} \frac{\theta_q\left(\frac{q^{\alpha+1}}{(1-q)} \frac{x}{\lambda}\right)}{\theta_q\left(\frac{q}{(1-q)} \frac{x}{\lambda}\right)} \sum_{n \geq 0} \frac{(1-q)^n}{(q; q)_n} \left(\frac{1}{q^\alpha x}\right)^n \\ &= \frac{(q; q)_\infty}{(q^{1-\alpha}; q)_\infty} (1-q)^\alpha \frac{\theta_q(q^\alpha\lambda)}{\theta_q(\lambda)} \frac{\theta_q\left(\frac{q^{\alpha+1}}{(1-q)} \frac{x}{\lambda}\right)}{\theta_q\left(\frac{q}{(1-q)} \frac{x}{\lambda}\right)} (1-q)^{-\alpha} \sum_{n \geq 0} \frac{(1-q)^n}{(q; q)_n} \left(\frac{1}{q^\alpha x}\right)^n \\ &= \Gamma_q(1-\alpha) \frac{\theta_q(q^\alpha\lambda)}{\theta_q(\lambda)} \frac{\theta_q\left(\frac{q^{\alpha+1}}{(1-q)} \frac{x}{\lambda}\right)}{\theta_q\left(\frac{q}{(1-q)} \frac{x}{\lambda}\right)} (1-q)^{-\alpha} \sum_{n \geq 0} \frac{(1-q)^n}{(q; q)_n} \left(\frac{1}{q^\alpha x}\right)^n. \end{aligned} \quad (36)$$

By the limiting formula (19) in Proposition 1, we can also obtain the well-defined limit $q \rightarrow 1 - 0$ of (36),

$$\lim_{q \rightarrow 1-0} \tilde{\psi}_\lambda^B \left(\frac{x}{1-q} \right) = \Gamma(1-\alpha) \times (\lambda)^{-\alpha} \times \left(\frac{x}{\lambda} \right)^{-\alpha} \sum_{n \geq 0} \frac{1}{n!} \left(\frac{1}{x} \right)^n = \Gamma(1-\alpha) x^{-\alpha} e^{\frac{1}{x}}.$$

Therefore, we obtain the limit $q \rightarrow 1 - 0$ of our new formulae $\tilde{\psi}_\lambda^B(x)$. \square

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